Rouquier (joint work with Joe CHUANG)

$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ $\quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[e, f] = h$$

$V_1 = \mathbb{C}^2 \circlearrowleft \mathfrak{g}$ $\qquad V_n = S^n \mathbb{C}^2$

1) Every finite dimensional rep. of $\mathfrak{g}$ is semisimple
(locally finite)

2) $\{V_n\}_{n \geq 0}$: all irreducible finite dimensional rep.

k : alg. closed field, $q \in k \setminus \{0, 1\}$

n $\quad H_n^f$: Hecke algebra of S_n

"$k\langle T_1, \ldots, T_n \rangle$ $\qquad (T_i + 1)(T_i - q) = 0$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_i T_j = T_j T_i \qquad |i-j| > 1$$

H_n : affine Hecke algebra

$= \underbrace{k[X_1^{\pm}, \ldots, X_n^{\pm}]}_{\text{subalg}} \otimes \underbrace{H_n^f}_{\text{subalg}}$ as vector space

$P \in k[X_1^{\pm}, \ldots, X_n^{\pm}]$ $\qquad T_i P - s_i(P) T_i = (q-1) \dfrac{P - s_i(P)}{1 - \cancel{X_i} X_{i+1}^{-1}}$

$s_i = (i \; i+1)$

Fix $a \in k^{\times}$

Let $I_n = \sum_{i=1}^{n} k[X_1^{\pm}, \ldots, X_n^{\pm}](X_i - a) \cap k[X_1^{\pm}, \ldots, X_n^{\pm}]^{S_n}$

$$\overline{H}_n := H_n / H_n I_n H_n = \underbrace{k[X_1^{\pm}, \ldots, X_n^{\pm}] / I_n \, k[X_1^{\pm} \ldots X_n^{\pm}]}_{\dim = I_n} \otimes H_n^f$$

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Fix n

Let $0 \leqq i \leqq n$ Put A_i = image of H_i in $\overline{H}_n$

$$= (\cdots) \otimes H_i^f$$

$$\Vert$$

$$k[X_1^{\pm} \ldots X_i^{\pm}]/ \ldots$$

$n=1$ $A_0 = A_1 = k$

$n=2$ $A_0 = k$, $A_1 = k[X_1^{\pm}]/(X_1^2 - a^2)$, $A_2 = k\langle X_1, T_1 \mid X_1^2 = a^2,\ (T_1+1)(T_1 - q) = 0,\ X_1 T_1 = (T+1-q)(2a - X_1) \rangle$

$$A_2 \cong M_2(k)$$

$$T_1 \mapsto \begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}$$

$$X_1 \mapsto \begin{pmatrix} a & n(1-q) \\ 0 & a \end{pmatrix}$$

General : $\overline{H}_n = A_n \cong M_{n!}(k)$ $(q \neq 1)$

(blocks of cyclotomic Hecke algebra)
Ariki-Koike

<u>Fact</u>. $A_i = M_{i!}(Z(A_i))$ $Z(A_i) = H^*(Gr(n,i), k)$

unique simple module S_i $\dim S_i = i!$

$$\dim A_i = \frac{n!}{(n-i)!} i!$$

$$\begin{array}{ccc} & A_n\text{-mod} & \\ \text{Ind}\uparrow & & \downarrow\text{Res} \\ & A_{n-1}\text{-mod} & \\ \vdots & & \downarrow \\ \text{Ind}\uparrow & & \downarrow\text{Res} \\ & A_1\text{-mod} & \\ \text{Ind}\uparrow & & \downarrow\text{Res} \\ & A_0\text{-mod} & \end{array}
\qquad
\begin{array}{lcl} & \oplus A_i\text{-mod} & \\ E: & \uparrow & = \text{Ind} \\ & \oplus A_i\text{-mod} & \\ F & \downarrow & = \text{Res} \end{array}$$

$$\begin{cases} [E\, S_i] = (n-i)[S_{i+1}] \\ [F\, S_i] = i\,[S_{i-1}] \end{cases}$$

$$\therefore\ [EFS_i] - [FES_i] = (2i-n)[S_i]$$

$\therefore\ \bigoplus_{i=0}^{n} K_0(A_i\text{-mod}) \otimes \mathbb{C}$: rep. of $\mathfrak{sl}_2(\mathbb{C})$

$e = [E]$, $f = [F]$

basis given by $[S_i]$ $\quad$ h acts by $2i-n$ on $K_0(A_i)$

this is V_n

Ex, $n=2$ (Good)

$$\begin{array}{ccc} \text{Res}\uparrow & k\text{-mod} \cong A_2\text{-mod} & \downarrow\text{Ind} \\ & k[X]/(X^2)\text{-mod} & \\ \text{Ind}\uparrow & & \downarrow\text{Res} \\ & k\text{-mod} & \end{array}
\qquad
\begin{array}{c} \mathbb{C} \\ 1\uparrow\downarrow 2 \\ \mathbb{C} \\ 2\uparrow\downarrow 1 \\ \mathbb{C} \end{array}$$

Ex,

$$\begin{array}{ccc} & k[X]/(X^2)\text{-mod} & \\ \text{Ind} & \uparrow\downarrow & \text{Res} \\ & k\text{-mod} & \\ \text{Res} & \uparrow\downarrow & \text{Ind} \\ & k[X]/(X^2)\text{-mod} & \end{array}
\qquad
\begin{array}{c} \mathbb{C} \\ 2\uparrow\downarrow 1 \\ \mathbb{C} \\ 1\uparrow\downarrow 2 \\ \mathbb{C} \end{array}
\qquad \text{Get } V_2 \circlearrowleft \mathfrak{sl}_2\mathbb{C}$$

bad example

no affine Hecke

no minimal characterization

Basic principle of category theory :

object = bad

maps = good

" $\quad$ 2-cat :

category = very bad

functors = bad

natural transf of functors = good

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$\Rightarrow$ Have to bring <u>natural</u> <u>transf.</u> of functors in

<u>Definition</u>

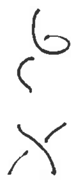
$\mathcal{A}$: abelian category $/ \Bbbk$ where every object has a finite comp. series

An <u>$\mathfrak{sl}_2$-categorification of $\mathcal{A}$</u> is the <u>data</u> of

- E, F : exact functor $\mathcal{A} \to \mathcal{A}$
- (E, F) : adjoint pair $\eta : 1 \to FE$ $\varepsilon : EF \to 1$

Conditions
- * $e = [E]$, $f = [F]$ gives a locally finite representation of $\mathfrak{sl}_2(\mathbb{C})$ on $V = \mathbb{C} \otimes K_0(\mathcal{A})$
- * $\forall S \in \mathcal{A}$ simple $[S]$ is a weight vector.
- * F : isomorphic to left adjoint of E

Also
- * $X \in \operatorname{End} E$, $T \in \operatorname{End} E^2$
- * $q \in \Bbbk \setminus \{0, 1\}$, $a \in \Bbbk^*$



$\stackrel{?}{\approx} X$

$= T$

*
$$\begin{tikzcd}[column sep=small]
 & E^3 \arrow[dl, "T1_E"'] \arrow[dr, "1_ET"] & \\
E^3 \arrow[d, "1_ET"'] & 2 & E^3 \arrow[d, "T1_E"] \\
E^3 \arrow[dr, "T1_E"'] & & E^3 \arrow[dl, "1_ET"] \\
 & E^3 &
\end{tikzcd}$$

- * $X - a$ locally nilpotent
- * $T \circ (1_E X) \circ T = q(X \cdot 1_E)$ in $\operatorname{End} E^2$

Fact $\mathcal{V}(n) = \bigoplus_{i=0}^{n} A_i\text{-mod}$ can be given the structure of an $\mathfrak{sl}_2$-categorification

In E $\quad A_{i+1} \otimes_{A_i} -$ defines X

(X_{i+1} acts on it; A_{i+1} acts on it)

In E^2 $\quad A_{i+2} \otimes_{A_i} -$ defines T

(T_{i+1} acts on it; A_{i+2} acts on it)

In the bad example, we do not have such a structure.

THM1

$\mathcal{A}$: $\mathfrak{sl}_2$-categorification $\qquad V = \bigoplus_{\lambda} V_\lambda$

define $\mathcal{A}_\lambda = \{ N \in \mathcal{A} \mid [N] \in V_\lambda \}$

$V^{\leq n}$ = sum of irr. subrep. of dim $\leq n+1$

$\mathcal{A}^{\leq n} = \{ N \in \mathcal{A} \mid [N] \in V^{\leq n} \}$

1) $\mathcal{A} = \bigoplus \mathcal{A}_\lambda$

2) $0 = \mathcal{A}^{\leq -1} \subset \mathcal{A}^{\leq 0} \subset \mathcal{A}^{\leq 1} \subset \cdots$ filtration by Serre sub-$\mathfrak{sl}_2$-categ.

$\bigcup_n \mathcal{A}^{\leq n} = \mathcal{A}$

and $\mathcal{A}^{\leq n} / \mathcal{A}^{\leq n-1}$ is an $\mathfrak{sl}_2$-categorification whose K_0 is a multiple of V_n

3) If V is a multiple of V_n, then

$\mathcal{A} \cong \mathcal{M} \otimes \mathcal{V}(n)$ $\qquad \mathcal{M}$: multiplicity abelian category

E, F acts trivially

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Thm 2 Fix $\lambda \geq 0$

$$EF|_{\mathcal{A}_{-\lambda}} \oplus \mathrm{Id}|_{\mathcal{A}_{-\lambda}}^{\oplus \lambda} \xrightarrow{\sim} FE|_{\mathcal{A}_{-\lambda}}$$

$$\sigma + \sum_{j=0}^{\lambda-1} (1_F X^j) \circ \eta$$

$$\sigma : EF \xrightarrow{\eta EF} FEEF \xrightarrow{FTF} FEEF \xrightarrow[FE\varepsilon]{} FE$$

Also

$$D^b(\mathcal{A}_\lambda) \xrightarrow{\sim} D^b(\mathcal{A}_{-\lambda})$$